

# On groups of diffeomorphisms of the interval with finitely many fixed points I

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**Abstract:** We strengthen the results of [1], consequently, we improve the claims of [2] obtaining the best possible results. Namely, we prove that if a subgroup  $\Gamma$  of  $\text{Diff}_+(I)$  contains a free semigroup on two generators then  $\Gamma$  is not  $C_0$ -discrete. Using this we extend the Hölder's Theorem in  $\text{Diff}_+(I)$  classifying all subgroups where every non-identity element has at most  $N$  fixed points. By using the concept of semi-archimedean groups, we also show that the classification picture fails in the continuous category.

## 1. INTRODUCTION

Throughout this paper we will write  $\Phi$  (resp.  $\Phi^{\text{diff}}$ ) to denote the class of subgroups of  $\Gamma \leq \text{Homeo}_+(I)$  (resp.  $\Gamma \leq \text{Diff}_+(I)$ ) such that every non-identity element of  $\Gamma$  has finitely many fixed points. Let us point out immediately that any subgroup of  $\text{Diff}_+^\omega(I)$  - the group of orientation preserving analytic diffeomorphisms of  $I$  - belongs to  $\Phi$ . In fact, many of the major algebraic and dynamical properties of subgroups of  $\text{Diff}_+^\omega(I)$  is obtained solely based on this particular property of analytic diffeomorphisms having only finitely many fixed points. Interestingly, groups in  $\Phi$  may still have both algebraic and dynamical properties not shared by any subgroup of  $\text{Diff}_+^\omega(I)$ . In particular, not every group in  $\Phi$  is conjugate to a subgroup of  $\text{Diff}_+^\omega(I)$ .

For a non-negative integer  $N \geq 0$ , we will also write  $\Phi_N$  (resp.  $\Phi_N^{\text{diff}}$ ) to denote the class of subgroups of  $\Gamma \leq \text{Homeo}_+(I)$  (resp.  $\Gamma \leq \text{Diff}_+(I)$ ) such that every non-identity element of  $\Gamma$  has at most  $N$  fixed points in the interval  $(0, 1)$ .

Characterizing  $\Phi_N$  for an arbitrary  $N$  is a major open problem solved only for values  $N = 0$  and  $N = 1$ : Hölder's Theorem states that any subgroup of  $\Phi_0$  is Abelian, while Solodov's Theorem states <sup>1</sup> that any subgroup of  $\Phi_1$  is metaabelian, in fact, it is isomorphic to subgroup of  $\text{Aff}_+(\mathbb{R})$  - the group of orientation preserving affine homeomorphisms of  $\mathbb{R}$ .

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<sup>1</sup>Solodov's result is unpublished but three independent proofs have been given by Barbot [3], Kovacevic [6], and Farb-Franks [4]

It has been proved in [1] that, for  $N \geq 2$ , any subgroup of  $\Phi_N^{\text{diff}}$  of regularity  $C^{1+\epsilon}$  is indeed solvable, moreover, in the regularity  $C^2$  we can claim that it is metaabelian. The argument there fails short in complete characterization of subgroups of  $\Phi_N^{\text{diff}}$ ,  $N \geq 2$  even at these increased regularities.

In [7], Navas gives a different proof of this result for groups of analytic diffeomorphisms, namely, it is shown that any group in  $\Phi_N^{\text{diff}}$  of class  $C^\omega$  is necessarily metaabelian.

In this paper, we provide a complete characterization of the class  $\Phi_N^{\text{diff}}$  for an arbitrary  $N$ . Our main result is the following

**Theorem 1.1.** *Let  $\Gamma \leq \text{Diff}_+(I)$  be an irreducible subgroup, and  $N \geq 0$  such that every non-identity element has at most  $N$  fixed points. Then  $\Gamma$  is isomorphic to a subgroup of  $\text{Aff}_+(\mathbb{R})$ .*

In other words, any irreducible subgroup of  $\Phi_N^{\text{diff}}$  is isomorphic to an affine group. Indeed, we show that, for  $N \geq 2$ , any irreducible subgroup of  $\Phi_N^{\text{diff}}$  indeed belongs to  $\Phi_1^{\text{diff}}$ ! Let us point out that there exist metaabelian examples (communicated to the author by A.Navas; a certain non-standard representation of the Baumslag-Solitar group  $BS(1, 2)$  in  $\text{Homeo}_+(I)$ ) which shows that the class  $\Phi_N$  is indeed strictly larger than the class  $\Phi_1$ , for  $N \geq 2$ . We will present examples of non-metabelian groups from  $\Phi_N$ ,  $N \geq 2$  thus constructing examples from  $\Phi_N$ ,  $N \geq 2$  which are algebraically non-isomorphic to subgroups of  $\Phi_1$ .

## 2. $C_0$ -DISCRETE SUBGROUPS OF $\text{Diff}_+(I)$ : STRENGTHENING THE RESULTS OF [1]

The main results of [2] are obtained by using Theorems B-C from [1]. Theorem B (Theorem C) states that a non-solvable (non-metabelian) subgroup of  $\text{Diff}_+^{1+\epsilon}(I)$  (of  $\text{Diff}_+^2(I)$ ) is non-discrete in  $C_0$  metric. Existence of  $C_0$ -small elements in a group provides effective tools in tackling the problem. Theorems B-C are obtained by combining Theorem A in [1] by the results of Szekeres, Plante-Thurston and Navas. Theorem A states that for a subgroup  $\Gamma \leq \text{Diff}_+(I)$ , if  $[\Gamma, \Gamma]$  contains a free semigroup in two generators then  $\Gamma$  is not  $C_0$ -discrete. In the proof of Theorem A, the hypothesis that the generators of the free semigroup belong to the commutator subgroup  $[\Gamma, \Gamma]$  is used only to deduce that the derivatives of both of the generators at either of the end points of the interval  $I$  equal 1. Thus we have indeed proved the following claim:

Let  $\Gamma \leq \text{Diff}_+(I)$  be a subgroup containing a free semigroup in two generators  $f, g$  such that either  $f'(0) = g'(0) = 1$  or  $f'(1) = g'(1) = 1$ . Then  $\Gamma$  is not  $C_0$ -discrete, moreover, there exists non-identity elements in  $[\Gamma, \Gamma]$  arbitrarily close to the identity in  $C_0$  metric.

In this section, we make a simple observation which strengthens Theorem A further, namely, the condition “ $[\Gamma, \Gamma]$  contains a free semigroup” can be replaced altogether with “ $\Gamma$  contains a free semigroup” (i.e. without demanding the extra condition “either  $f'(0) = g'(0) = 1$  or  $f'(1) = g'(1) = 1$ ”).

**Theorem 2.1** (Theorem A'). *Let  $\Gamma \leq \text{Diff}_+(I)$  be a subgroup containing a free semigroup in two generators. Then  $\Gamma$  is not  $C_0$ -discrete, moreover, there exists non-identity elements in  $[\Gamma, \Gamma]$  arbitrarily close to the identity in  $C_0$  metric.*

In the proof of Theorems B-C, if we use Theorem A' instead of Theorem A we obtain the following stronger versions.

**Theorem 2.2** (Theorem B'). *If a subgroup  $\Gamma \leq \text{Diff}_+^{1+\epsilon}(I)$  is  $C_0$ -discrete then it is nilpotent.*

**Theorem 2.3** (Theorem C'). *If a subgroup  $\Gamma \leq \text{Diff}_+^2(I)$  is  $C_0$ -discrete then it is Abelian.*

Theorem A' is obtained from the proof of Theorem A by a very slight modification. Let us first assume that  $\Gamma$  is irreducible, i.e. it has no fixed point on  $(0, 1)$ . Let  $f, g \in \Gamma$  generate a free semigroup on two generators. If  $f'(0) = g'(0) = 1$  or  $f'(1) = g'(1) = 1$  then the claim is already proved in [1], otherwise, without loss of generality we may assume that  $f'(1) < 1$  and  $g'(1) < 1$ .

Let also  $\epsilon, N, \delta, M, \theta$  be as in the proof of Theorem A in [1], except we demand that  $1 < \theta_N < \sqrt[8N]{1.9}$  (instead of  $1 < \theta_N < \sqrt[2N]{2}$ ), and instead of the inequality  $\frac{1}{\theta_N} < \phi'(x) < \theta_N$ , we demand that

$$\max_{x, y \in [1-\delta, 1]} \left( \frac{\phi'(x)}{\phi'(y)} \right)^8 < \theta_N,$$

where  $\phi \in \{f, g, f^{-1}, g^{-1}\}$ . In addition, we also demand that for all  $x \in [1 - \delta, 1]$ , we have  $f(x) > x$  and  $g(x) > x$ .

Then we let  $W = W(f, g)$ ,  $\alpha, \beta \in \Gamma$  be as in the proof of Theorem A. We may also assume that (by replacing  $(\alpha, \beta)$  with  $(\alpha\beta, \beta\alpha)$  if necessary),  $\alpha'(0) = \beta'(0) = \lambda < 1$ .

Now, for every  $n \in \mathbb{N}$ , instead of the set

$$\mathbb{S}_n = \{U(\alpha, \beta)\beta\alpha \mid U(\alpha, \beta) \text{ is a positive word in } \alpha, \beta \text{ of length at most } n\}$$

we consider the set

$$\mathbb{S}'_n = \{U(\alpha, \beta)\beta\alpha \in \mathbb{S}_n \mid \text{sum of exponents of } \alpha \text{ in } U(\alpha, \beta) \text{ equals } \lfloor \frac{n}{2} \rfloor\}$$

Previously, we had the crucial inequality  $|\mathbb{S}_n| \geq 2^n$  for all  $n$  but now we have the inequality  $|\mathbb{S}'_n| \geq (1.9)^n$  for sufficiently big  $n$ . Let us also observe that, for any interval  $J$  in  $(1 - \delta, 1)$ , and for all  $g \in \mathbb{S}'_n$ , we will have the inequality  $|g(J)| < \lambda^n(\theta_N)^{\frac{1}{8}n}$ . Then for some sufficiently big  $n$  the following conditions hold:

(i) there exist  $g_1, g_2 \in \mathbb{S}_n$  such that  $g_1 \neq g_2$ , and

$$|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{2N\sqrt[2N]{1.9}^n}, 1 \leq i \leq N-1,$$

(ii)  $M^{2m+4}(\theta_N)^{4n} \frac{1}{2N\sqrt[2N]{1.9}^n} < \epsilon$ ,

where  $x_i = \frac{i}{N}, 0 \leq i \leq N$ .

The rest of the proof goes exactly the same way by replacing  $\mathbb{S}_n$  with  $\mathbb{S}'_n$ : letting again  $h_1 = g_1 W, h_2 = g_2 W$ , we obtain that  $|h_1^{-1} h_2(x) - x| < 2\epsilon$  for all  $x \in [0, 1]$ . Since  $\epsilon$  is arbitrary, we obtain that  $\Gamma$  is not  $C_0$ -discrete. On the other hand, by definition of  $\mathbb{S}'_n$  we have  $h_1^{-1} h_2 \in [\Gamma, \Gamma]$ . If  $\Gamma$  is not irreducible then it suffices to observe that there exists only finitely many intervals  $I_1, \dots, I_m$  in  $(0, 1)$  such that  $\Gamma$  fixes the endpoints of  $I_j$  but no other point inside  $I_j$ , moreover,  $\sum_{1 \leq j \leq m} |I_j| > 1 - 2\epsilon$

□

### 3. EXTENSION OF HÖLDER'S THEOREM IN $\text{Diff}_+(I)$

Let us point out that the following theorem follows from the proof of Theorem 0.1 and Theorem 0.2 in [2].

**Theorem 3.1.** *Let  $N \geq 0$  and  $\Gamma$  be an irreducible group in  $\Phi_N^{\text{diff}}$  such that  $[\Gamma, \Gamma]$  contains diffeomorphisms arbitrarily close to the identity in  $C_0$  metric. Then  $\Gamma$  belongs to  $\Phi_1^{\text{diff}}$  thus it is isomorphic to a subgroup of the affine group  $\text{Aff}_+(\mathbb{R})$ .*

The method of [2] does not allow to obtain a complete classification of subgroups of  $\Phi_N^{\text{diff}}$  primarily because existence of non-discrete subgroups in  $\text{Diff}_+^{1+\epsilon}(I)$  (in  $\text{Diff}_+^2(I)$ ) is guaranteed only for non-solvable (non-metaabelian) groups. Within the class of solvable (metaabelian) groups the method is inapplicable.

Now, by Theorem A', we can guarantee the existence of non-discreteness in the presence of a free semigroup. On the other hand, the property of containing a free semigroup on two generators is generic only in  $C^{1+\epsilon}$  regularity; more precisely, any non-virtually nilpotent subgroup of  $\text{Diff}_+^{1+\epsilon}(I)$  contains a free semigroup on two generators. Just in  $C^1$ -regularity,  $\text{Diff}_+(I)$  has many non-virtually nilpotent subgroups (e.g. subgroups of intermediate growth) without free semigroups. (see [8])

The next proposition indicates a strong distinctive feature for groups of  $\Phi$ , and supplies free semigroups for all non-Abelian subgroups in  $\Phi_N, N \geq 1$ .

**Proposition 3.2.** *Any subgroup in class  $\Phi$  is either Abelian or contains a free semigroup on two generators.*

**Corollary 3.3.** For any  $N \geq 0$ , a subgroup of  $\Phi_N$  is either Abelian or contains a free semigroup.

**Remark 3.4.** Let us point out that any group  $\Gamma$  in  $\Phi$  is bi-orderable. A bi-order can be given as follows: for  $f, g \in \Gamma$ , we let  $f < g$  iff  $f(x) < g(x)$  in some interval  $(0, \delta)$ . Proposition 3.2 shows that the converse is far from being true, i.e. not every finitely generated bi-orderable group embeds in  $\Phi$ . For example, it is well known that every finitely generated torsion-free nilpotent group is bi-orderable hence it embeds in  $\text{Homeo}_+(I)$  (by the result of [5] it embeds into  $\text{Diff}_+(I)$  as well); on the other hand, a finitely generated nilpotent group does not contain a free semigroup on two generators.

We need the following well known notion.

**Definition 3.5.** Let  $f, g \in \text{Homeo}_+(I)$ . We say the pair  $(f, g)$  is *crossed* if there exists a non-empty open interval  $(a, b) \subset (0, 1)$  such that one of the homeomorphisms fixes  $a$  and  $b$  but no other point in  $(a, b)$  while the other homeomorphism maps either  $a$  or  $b$  into  $(a, b)$ .

It is a well known folklore result that if  $(f, g)$  is a crossed pair then the subgroup generated by  $f$  and  $g$  contains a free semigroup on two generators (see [9]).

**Proof of Proposition 3.2.** We may assume that  $\Gamma$  is irreducible. If  $\Gamma$  acts freely then by Hölder's Theorem it is Abelian and we are done. Otherwise, there exists a point  $p \in (0, 1)$  which is fixed by some non-identity element  $f$  of  $\Gamma$ . Since  $\Gamma$  is not irreducible, there exists  $g$  which does not fix  $p$ . Let  $p_+$  be the biggest fixed point of  $g$  less than  $p$ , and  $p_-$  be the smallest fixed point of  $g$  bigger than  $p$ . If at least one of the points  $p_+, p_-$  is not fixed by  $f$  then either the pair  $(f, g)$  or  $(f^{-1}, g)$  is crossed.

Now assume that both  $p_+, p_-$  are fixed by  $f$ . Without loss of generality we may also assume that  $g(x) > x$  for all  $x \in (p_+, p_-)$ . Let  $q_-$  be the smallest fixed point of  $f$  bigger than  $p_-$ , and  $q_+$  be the biggest fixed point of  $f$  smaller than  $p_+$ . (we have  $q_- \leq q_+$  but it is possible that  $q_-$  equals  $q_+$ ). Then there exists  $n \geq 1$  such that  $g^n(q_-) > q_+$ . Then either the pair  $(g^n f g^{-n}, f)$  or the pair  $(g^n f^{-1} g^{-n}, f)$  is crossed (in the interval  $(a, b) = (q_+, p_+)$ ).  $\square$

#### 4. SEMI-ARCHIMEDIAN GROUPS

It is a well known fact that any subgroup of  $\text{Homeo}_+(\mathbb{R})$  is left-orderable. Conversely, one can realize any countable left-orderable group as a subgroup of  $\text{Homeo}_+(\mathbb{R})$  (see [9]). Despite such an almost complete and extremely useful characterization of left-orderable groups, when presented algebraically (or otherwise) it can be difficult to decide if the group does admit a left order at all, and if yes, then are there many left orders?

For example, it is true that a semi-direct product of a left-orderable group with another left-orderable group is still left-orderable. In fact, if the groups  $G, H$  admit left orders  $\prec_1, \prec_2$  respectively, then one can put a left order  $\prec$  on  $G \ltimes H$  by letting,  $(g_1, h_1) \prec (g_2, h_2)$  iff either  $g_1 \prec_1 g_2$  or  $g_1 = g_2, h_1 \prec_2 h_2$ . This left order is quite straightforward; here,  $G$  is dominant over  $H$  and because  $G$  is the acting group, one checks directly that the linear order  $\prec$  is indeed left-invariant. It is sometimes more interesting (and needed for our purposes in this paper) to make  $H$  dominant over  $G$ ; one can do this if the action of  $G$  on  $H$  preserves the left order of  $H$ . We materialize this in the following

**Lemma 4.1.** *Let a group  $G_1$  acts on a group  $G_2$  by automorphisms. Let  $\prec_1, \prec_2$  be left orders on  $G_1, G_2$  respectively, and assume that the action of  $G_1$  on  $G_2$  preserves the left order [i.e. if  $g \in G_1, x_1, x_2 \in G_2, x_1 \prec_2 x_2$  then  $g(x_1) \prec_2 g(x_2)$ ].*

*Then there exists a left order  $<$  in  $G_1 \rtimes G_2$  which satisfies the following conditions:*

- 1) *if  $g_1, f_1 \in G_1, g_1 \prec_1 f_1$  then  $(g_1, 1) < (f_1, 1)$ ;*
- 2) *if  $g_2, f_2 \in G_2, g_2 \prec_2 f_2$  then  $(1, g_2) < (1, f_2)$ ;*
- 3) *if  $g_1 \in G_1 \setminus \{1\}, g_2 \in G_2 \setminus \{1\}, 1 \prec_2 g_2$ , then  $(g_1, 1) < (1, g_2)$ .*

**Proof.** We define the left order on  $G_1 \rtimes G_2$  as follows: given  $(g_1, f_1), (g_2, f_2) \in G_1 \rtimes G_2$  we define  $(g_1, f_1) < (g_2, f_2)$  iff either  $f_1 \prec_2 f_2$  or  $f_1 = f_2, g_1 \prec_1 g_2$ . Then the claim is a direct check.  $\square$

The left order  $<$  on the semidirect product  $G_1 \rtimes G_2$  constructed in the proof of the lemma will be called the *extension of  $\prec_1$  and  $\prec_2$* .

Let  $G$  be a group with a left order  $<$ .  $G$  is called *Archimedean* if for any two positive elements  $f, g \in G$ , there exists a natural number  $n$  such that  $g^n > f$ . In other words, for any positive element  $f$ , the sequence  $(f^n)_{n \geq 1}$  is *strictly increasing* and *unbounded*.<sup>2</sup> It is a classical result, proved by Hölder, that Archimedean group are necessarily Abelian, moreover, they are always isomorphic to a subgroup of  $\mathbb{R}$ . In fact, the notion of Archimedean group arises very naturally in proving the fact that any freely acting subgroup of  $\text{Homeo}_+(\mathbb{R})$  is Abelian, first, by showing that such a group must be Archimedean, and then, by a purely algebraic argument (due to Hölder), proving that *Archimedean*  $\Rightarrow$  *Abelian*.

It turns out one can generalize the notion of Archimedean groups to obtain algebraic results of similar flavor for subgroups of  $\text{Homeo}_+(\mathbb{R})$  which do not necessarily act freely but every non-trivial element has at most  $N$  fixed points. Let us first consider the following property.

**Definition 4.2.** Let  $G$  be a group with a left order  $<$ . We say  $G$  satisfies property  $(P_1)$  if there exists a natural number  $M$  and elements  $g, \delta \in G$  such that if the sequence  $(g^n)_{n \geq 1}$  is increasing but bounded, and  $\delta g^k > g^m$  for all  $k, m > M$ , then for all  $k \geq M$  either the sequence  $(g^n \delta g^k)_{n \geq 1}$  or the sequence  $(g^{-n} \delta g^k)_{n \geq 1}$  is increasing and unbounded.

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<sup>2</sup>In a left-orderable group  $G$ , we say a sequence  $(g_n)_{n \geq 1}$  is bounded if there exists an element  $g$  such that  $g^{-1} < g_n < g$  for all  $n \geq 1$ .

Every Archimedean group clearly satisfies property  $(P_1)$  but there are non-archimedean groups too with property  $(P_1)$ . In fact, it is easy to verify that the metaabelian affine group  $\text{Aff}_+(\mathbb{R})$  with the following very natural order does satisfy property  $(P_1)$  while not being Archimedean: for any two maps  $f, g \in \text{Aff}_+(\mathbb{R})$  we say  $f < g$  iff either  $f(0) < g(0)$  or  $f(0) = g(0), f(1) < g(1)$ .

An Archimedean group can be viewed as groups where powers of positive elements *reach infinity*. In groups with property  $(P_1)$ , the power of a positive element reaches infinity perhaps after an extra *arbitrarily small one time push*, namely if  $g \in G$  is positive and  $(g^n)_{n \geq 1}$  is still bounded, then for every  $\delta$  where  $\delta g^m > g^k$  for all sufficiently big  $m, k$ , either the sequence  $g^n \delta g^m_{n \geq 1}$  or the sequence  $g^{-n} \delta g^m_{n \geq 1}$  reaches the infinity. Thus groups with property  $(P_1)$  can be viewed as generalization of Archimedean groups. We would like to introduce even a more general property  $(P_N)$  for any  $N \geq 1$ . (Archimedean groups can be viewed as exactly the groups with property  $(P_0)$ ).

**Definition 4.3.** Let  $G$  be a group with a left order  $<$ , and  $N$  be a natural number. We say  $G$  satisfies property  $(P_N)$  if there exists a natural number  $M$ , the elements  $g, \delta_1, \dots, \delta_{N-1} \in G$ , and the numbers  $\epsilon_1, \dots, \epsilon_{N-1} \in \{-1, 1\}$  such that if, for all  $i \in \{1, \dots, N-1\}$  and for all  $k_1, \dots, k_{i-1}, k_i \geq M, \epsilon_1, \dots, \epsilon_i \in \{-1, 1\}$ ,

(i) the sequence  $(g^{\epsilon_i n} \delta_{i-1} g^{\epsilon_{i-1} k_{i-1}} \dots \delta_1 g^{\epsilon_1 k_1})_{n \geq 1}$  is bounded from above, and

(ii)  $\delta_i g^{\epsilon_i k_i} \delta_{i-1} g^{\epsilon_{i-1} k_{i-1}} \dots \delta_1 g^{\epsilon_1 k_1} > g^{\epsilon_i k_i} \delta_{i-1} g^{\epsilon_{i-1} k_{i-1}} \dots \delta_1 g^{\epsilon_1 k_1}$

then, for some  $\epsilon_N \in \{-1, 1\}$ , the sequence  $(g^{\epsilon_N n} \delta_{N-1} g^{\epsilon_{N-1} k_{N-1}} \dots \delta_1 g^{\epsilon_1 k_1})_{n \geq 1}$  is unbounded from above.

**Remark 4.4.** Similarly, in groups with property  $(P_N)$  the power of a positive element may not necessarily reach the infinity but does so after some  $N$  arbitrarily small pushes (by  $\delta_1, \dots, \delta_N$ ). Namely, one considers the sequences  $g^n, g^{\pm n} \delta_1 g^n, g^{\pm n} \delta_2 g^{\pm n} \delta_1 g^n, \dots, g^{\pm n} \delta_N \dots g^{\pm n} \delta_1 g^n$  and one of them reaches infinity as  $n \rightarrow \infty$ .

**Remark 4.5.** In the case of  $N = 0$ , the existence of elements  $g_1, \delta_1, \dots, g_{N-1}, \delta_{N-1}$  is a void condition, and one can state condition  $(P_0)$  as the existence of an element  $g_0$  such that  $g_0^n$  is unbounded; thus groups with property  $(P_0)$  are exactly the Archimedean groups.

**Definition 4.6.** A left ordered group  $G$  is called *semi-Archimedean* if it satisfies property  $(P_N)$  for some  $N \geq 0$ .



We will need the following result about semi-Archimedean groups:

**Proposition 4.7.** *Let  $G$  be a countable semi-Archimedean group. Then  $G$  has a realization as a subgroup of  $\text{Homeo}_+(\mathbb{R})$  such that every non-identity element has at most  $N$  fixed points.*

**Proof.** For simplicity, we will first prove the proposition for  $N = 1$ . (In fact, for the application in the next section, Proposition 4.7 is needed only in the case  $N = 1$ ).

If there exists a smallest positive element in  $\Gamma$  then, necessarily,  $\Gamma$  is cyclic and the claim is obvious. Let  $g_1, g_2, \dots$  be all elements of  $\Gamma$  where  $g_1 = 1$ . We can embed  $\Gamma$  in  $\text{Homeo}_+(\mathbb{R})$  such that the sequence  $\{g_n(0)\}_{n \geq 1}$  satisfies the following condition:  $g_1(0) = 0$ , and for all  $n \geq 1$ ,

(i) if  $g_{n+1} > g_i$  for all  $1 \leq i \leq n$ , then  $g_{n+1}(0) = \max\{g_i(0) \mid 1 \leq i \leq n\} + 1$ ,

(ii) if  $g_{n+1} < g_i$  for all  $1 \leq i \leq n$ , then  $g_{n+1}(0) = \min\{g_i(0) \mid 1 \leq i \leq n\} - 1$ ,

(iii) if  $g_i < g_{n+1} < g_j$  and none of the elements  $g_1, \dots, g_n$  is strictly in between  $g_i$  and  $g_j$  then  $g_{n+1}(0) = \frac{g_i(0) + g_j(0)}{2}$ .

Then, since there is no smallest positive element in  $\Gamma$ , we obtain that the orbit  $O = \{g_n(0)\}_{n \geq 1}$  is dense in  $\mathbb{R}$ . This also implies that the group  $\Gamma$  for any point  $p \in O$  and for any open non-empty interval  $I$ , there exists  $\gamma \in \Gamma$  such that  $\gamma(p) \in I$ .

Now assume that some element  $g$  of  $\Gamma$  has at least two fixed points. Then for some  $p, q$  we have  $\text{Fix}(g) \cap [p, q] = \{p, q\}$ . Without loss of generality, we may also assume that  $p > 0$  and  $g(x) > x$  for all  $x \in (p, q)$ . By density of the orbit  $\{g_n(0)\}_{n \geq 1}$ , there exists  $f \in \Gamma$  such that  $f(0) \in (p, q)$ . Then, for sufficiently big  $n$ , we have  $\delta = g^{-n}f$  has a fixed point  $r \in (p, q)$ , moreover,  $\delta(x) > x$  for all  $x \in (p, r)$ .

Then  $g^{\epsilon n}$  does not reach infinity for any  $\epsilon \in \{-1, 1\}$ , in fact,  $g^{\epsilon n}(0) < p$  for all  $n \geq 1, \epsilon \in \{-1, 1\}$ . Then  $\{g^{\epsilon_1 n} \delta g^{\epsilon_k}\}_{n \geq 1}$  does not reach infinity for any  $k \in \mathbb{Z}, \epsilon, \epsilon_1 \in \{-1, 1\}$ . Contradiction.

To treat the case of general  $N \geq 1$ , let us assume that some element  $g \in \Gamma$  has at least  $N + 1$  fixed points. Then there exists open intervals  $I_1 = (a_1, b_1), \dots, I_{N+1} = (a_{N+1}, b_{N+1})$  such that  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_N < b_N \leq a_{N+1} < b_{N+1}$  and  $\{a_1, b_1, \dots, a_{N+1}, b_{N+1}\} \subset \text{Fix}(g)$ . By density of the orbit  $O$ , there exist elements  $\delta_1, \dots, \delta_N$  such that  $\delta_i(b_i) \in I_{i+1}, 1 \leq i \leq N$ . Then for the appropriate choices of  $\epsilon_1, \dots, \epsilon_{N-1} \in \{-1, 1\}$  and for sufficiently big  $k_1, \dots, k_{N_1}$ , conditions (i)

and (ii) of Definition 4.3 hold, while for any  $\epsilon_N \in \{-1, 1\}$ , the sequence  $(g^{\epsilon_N n} \delta_{N-1} g^{\epsilon_{N-1} k_{N-1}} \dots \delta_1 g^{\epsilon_1 k_1})_{n \geq 1}$  is bounded from above because it lies in  $I_{N+1}$ .  $\square$

**Remark 4.8.** Let us emphasize that in this section we did not make an assumption that the groups belong to the class  $\Phi$ .

## 5. A NON-AFFINE SUBGROUP OF $\Phi_N$

In this section we will present an irreducible non-affine subgroup  $\Gamma$  from  $\Phi_N$  for  $N \geq 2$  thus showing that the classification result of Theorem 1.1 fails in the continuous category. The method for the construction suggests that one can obtain a solvable group of arbitrarily high derived length in  $\Phi_N$  but we would like to emphasize that we do not know any non-solvable example.

The subgroup  $\Gamma$  will be given by a presentation

$$\langle t, s, b \mid tbt^{-1} = b^2, sb s^{-1} = b^2, [t, s] = 1 \rangle$$

so it has a relatively simple algebraic structure; it is indeed isomorphic to the semidirect product of  $\mathbb{Z}^2$  with the additive group of the ring  $D = \mathbb{Z}[\frac{1}{2}]$  where  $b$  can be identified with identity of the ring  $\mathbb{Z}[\frac{1}{2}]$ ,  $s, t$  can be identified with the standard generators of  $\mathbb{Z}^2$ , and the action of both  $t$  and  $s$  on  $D$  is by multiplication by 2. However, we will put a left order in it which is not the most natural left order that one considers.

Let  $\prec_1$  be the natural left order on  $\mathbb{Z}$ , and  $\prec_2$  be the left order on  $\mathbb{Z}[\frac{1}{2}]$  induced by the usual order on  $\mathbb{R}$ . Notice that the action of  $\mathbb{Z}$  on the group  $\mathbb{Z}[\frac{1}{2}]$  preserves the left order  $\prec_2$ . Then we let  $<$  be the extension of the left orders  $\prec_1$  and  $\prec_2$ . By Lemma 4.1,  $<$  is a left order on  $\Gamma$ . One can check easily that the group  $\Gamma$  satisfies property  $(P_1)$ .

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